Lecture 13: Generative Adversarial Networks - I

COMP 5801H/4900A: Generative AI and LLMs

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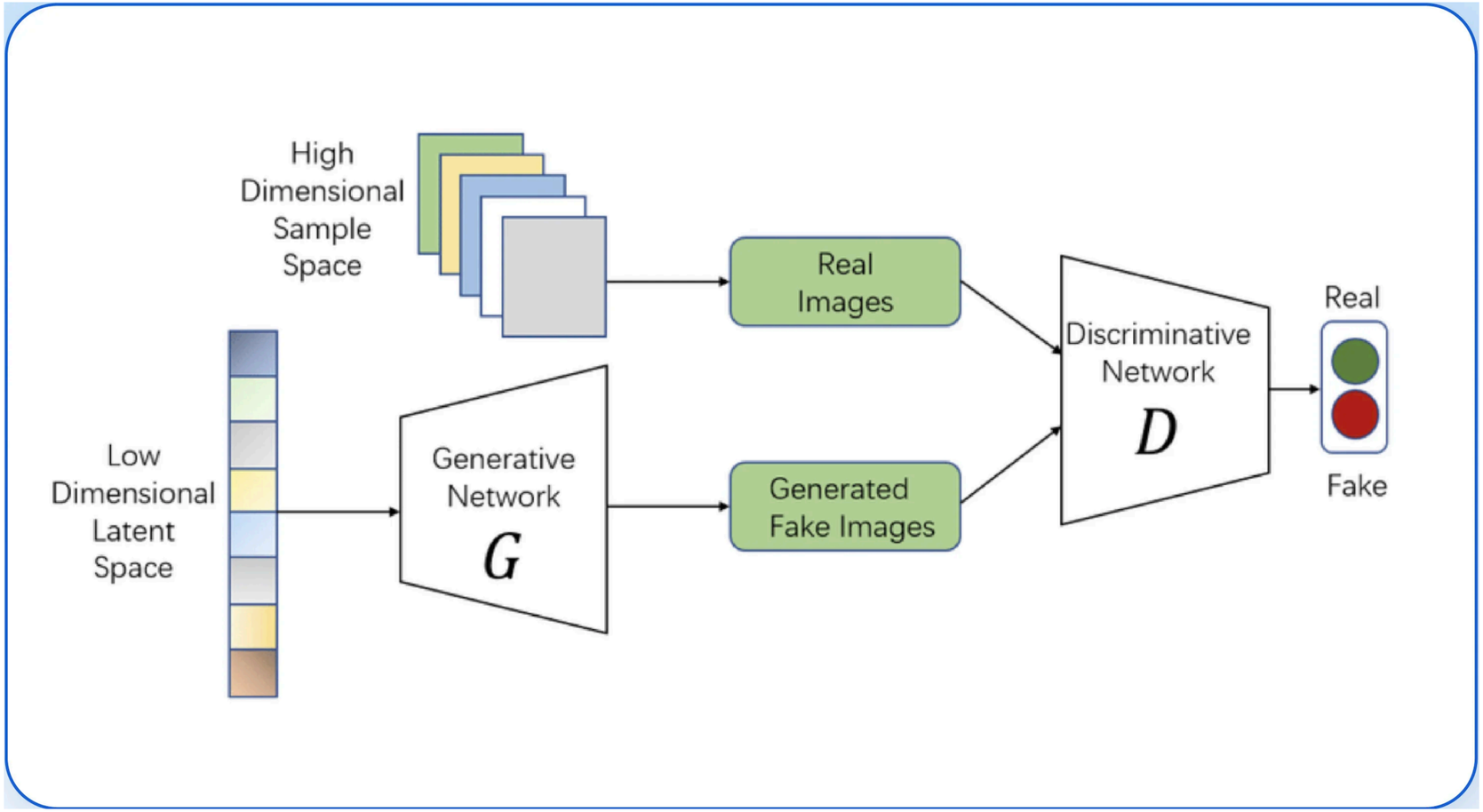
Outline

- Module 1: The Adversarial Concept
- Module 2: The Math of the Minimax Game
- Module 3: Training Dynamics & Challenges
- Module 4: Architectural Evolution
- Module 5: Comparison & Future

Generative Adversarial Networks (GANs)



Credit: https://www.xenonstack.com/blog/gans-for-image-synthesis

Generative Adversarial Networks (GANs)

- **The Paradigm Shift**: Moving from Likelihood Maximization (VAEs) to Game Theory (GANs).
- **Zero-Sum Game**: One network wins only if the other loses.
- **The Goal**: To reach "Nash Equilibrium"—the point where the Generator is so good the Discriminator can only guess with 50% accuracy.

The Core Philosophy: Sampling vs. Optimization

- **Explicit vs. Implicit Density**:
  - VAEs (Explicit): We define a mathematical distribution (the Gaussian) and force the data into it. We optimize a lower bound (ELBO).
  - GANs (Implicit): We don't care about the mathematical formula for the density. We just want to learn a procedure that produces samples that "look" like the data.
- **The Loss of "Distance"**:
  - In VAEs, if a pixel is off, the MSE/BCE loss increases.
  - In GANs, the "Loss" is just the opinion of another network. If the Discriminator says it looks real, it is successful, regardless of pixel-by-pixel accuracy.
- **The Sharpness Trade-off**:
  - VAEs tend to be "blurry" because they average over the Gaussian space.
  - GANs produce "sharp" results because they only need to be realistic enough to fool the Discriminator

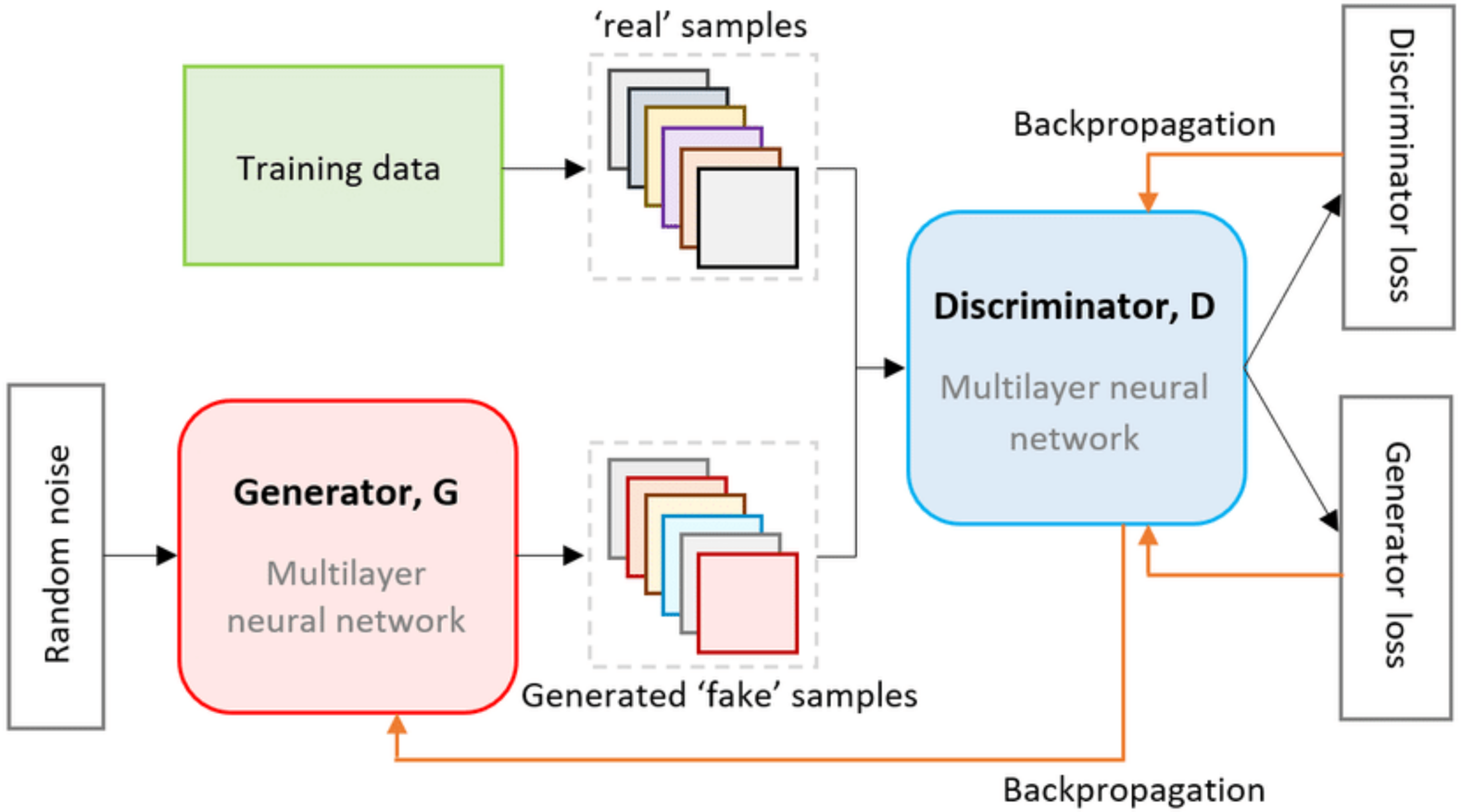
The Forger vs. The Detective

- **The Generator (G)**:
  - Role: The Counterfeiter.
  - Input: Random noise (z) from a simple distribution (e.g., Gaussian).
  - Goal: Map that noise into a complex data distribution (images) that is indistinguishable from the real thing.
  - It never sees the real images; it only hears the Detective's feedback.
- **The Discriminator (D)**:
  - Role: The Art Critic or Detective.
  - Input: Either a Real image (x) or a Fake image ($G(z)$).
  - Goal: Correcting classify "Real" as 1 and "Fake" as 0.
  - It is the teacher that the Generator must outsmart.

The "Co-evolution" Concept

- If the Detective is too lazy, the Forger stays bad.
- If the Detective is too perfect, the Forger gives up.
- Success requires both to get better at the exact same pace.

Architecture



Credit: Little et al. (2021) "Generative Adversarial Networks for Synthetic Data Generation: A Comparative Study"

The Two-Player Game: A Zero-Sum Framework

- **A Zero-Sum Game**: In game theory, a zero-sum game means one player's gain is exactly equal to the other player's loss.
- **The Objectives**:
  - **The Discriminator** (D): Wants to maximize its accuracy. It wants to be right every time.
  - **The Generator** (G): Wants to maximize the Discriminator's error rate. It wins when the Detective is fooled.
- **Sequential Optimization**: Unlike a standard neural network where we minimize one loss, here we have a "**Minimax**" problem. We are trying to find a point of stability where neither player can improve their position (Nash Equilibrium).

Why "Adversarial"?

- In VAEs, the Encoder and Decoder work together to reconstruct the image (Cooperation).
- In GANs, G and D are in **direct conflict**.
- **The Result**: This conflict prevents the model from becoming "lazy" (like the blurriness in VAEs). If G produces a blurry image, D will easily flag it as "Fake," forcing G to sharpen up to survive.

Defining the Generator (G)

- **The Input (z)**: * Typically sampled from a Standard Normal $\mathcal{N}(0,1)$
    - This represents the "seed" of the image. Just like in VAEs, different points in z will result in different generated images.
- **The Transformation**:
    - The Generator is a **differentiable function** $G(z)$.
    - It uses learned weights to "unflatten" the noise, gradually adding spatial structure (lines, then shapes, then textures).
- **Final Activation**:
    - Usually uses **Tanh** (scaling pixels between -1 and 1) or **Sigmoid** (0 to 1).
    - **Crucial Point**: G never "sees" a real image. Its only source of truth is the gradient flowing back from the Discriminator.

Defining the Discriminator (D)

- **The Input:**
  - The Discriminator is a "dual-input" network (though it sees only one image at a time).
  - It receives **Real Samples** (x) from the dataset and **Fake Samples** ($G(z)$) from the Generator.
- **The Task:**
  - It performs **Binary Classification (**but produces a scalar value $\in [0,1]$**)**.
  - **Output** ≈ 1: "I am confident this is a real image from the training set."
  - **Output** ≈ 0: "I am confident this was created by the Generator."
- **The "Teacher" Role:**
  - When the Discriminator is correct, it penalizes the Generator.
  - When it is fooled, it provides the "Gradient" that tells the Generator how to improve.

System Components

- **The Balancing Act**: This is a dynamic system. Unlike a VAE, where the loss usually goes down smoothly, a GAN's "Total Loss" is hard to interpret because as G improves, D gets tougher.
- **Independent Updates**: We typically update D for one step, then G for one step. They take turns "learning" from the current state of the other.
- **The Competition is the Teacher**:
  - D learns by looking at x and $G(z)$.
  - G learns **indirectly** through D. G never sees x! It only "sees" x through the eyes of the Discriminator's criticism.

The Math of the Game: The Minimax Objective

- **The objective function**

$$\min_G \max_D V(D, G) = \mathbb{E}_{x \sim p_{data}(x)}[\log D(x)] + \mathbb{E}_{z \sim p_z(z)}[\log(1 - D(G(z)))]$$

- $D(x)$: Detective's score for real data (should be 1).
- $D(G(z))$: Detective's score for fake data (should be 0).
- $\min_G \max_D$: A game of tug-of-war

The Math of the Game: The Minimax Objective

- **The Max Player (Discriminator D):**
  - Wants to maximize the probability of assigning the correct label to both real and fake images.
  - It wants $D(x)$ to be high (close to 1) and $D(G(z))$ to be low (so $1 - D(G(z))$ is close to 1).
- **The Min Player (Generator G):**
  - Wants to minimize the Discriminator's ability to distinguish.
  - It only appears in the second term. It wants $D(G(z))$ to be high (fooling the critic), which makes $1 - D(G(z))$ small.

The Discriminator's Objective

- **Isolating the Math for D:**

When we train the Discriminator, we treat the Generator as a "frozen" entity and focus only on this part of the value function:

$$\max_D \mathscr{L}_D = \underbrace{\mathbb{E}_{x\sim p_{data}}[\log D(x)]}_{\text{Correctly Identify Real}} + \underbrace{\mathbb{E}_{z\sim p_z}[\log(1 - D(G(z)))]}_{\text{Correctly Identify Fake}}$$

- **Term 1**: If $D(x)$ is 1.0, $\log(1) = 0$ (Perfect). If $D(x)$ is 0.1, $\log(0.1)$ is a large negative number (Penalty!).
- **Term 2**: If $D(G(z))$ is 0.0, $\log(1 - 0) = 0$ (Perfect). If $D(G(z))$ is 0.9, $\log(1 - 0.9)$ is a large negative number (Penalty!).

The Generator’s Objective

- **Isolating the Math for G:**

  The Generator only cares about the second half of the equation (the part involving its fakes). However, there is a "practical" twist in how we calculate it:

  - **The Theoretical Objective**: $\min_G \mathbb{E}_{z \sim p_z}[\log(1 - D(G(z)))]$
    - This says: "Minimize the probability that the Detective is right about my fakes."
  - The Practical (Non-Saturating) Objective: $\max_G \mathbb{E}_{z \sim p_z}[\log D(G(z))]$
    - This says: "Maximize the probability that the Detective thinks my fakes are Real."

Why the "Practical" Switch?

- **The Gradient Problem**: Early in training, the Generator is terrible. The Discriminator easily identifies fakes ($D(G(z)) \approx 0$).

- In the theoretical version, the gradient is very **"flat"** here, so the Generator learns slowly.

- In the practical version, the gradient is very **"steep"** when $D(G(z))$ is near zero, giving the Generator a much-needed **"kick-start."**

Notation Reference

Symbol	Name	Description
x	Sample	A specific image (either real or fake).
p_{data}	Data Distribution	The actual distribution of real images in the world.
p_g	Model Distribution	The distribution of images the Generator is currently "hallucinating."
z	Latent Noise	The random input that gives the generator the variety to create different images.

Global Optimality: The Nash Equilibrium

- What happens if we have an infinitely powerful Generator and Discriminator?
- Optimal D: For a fixed G, the best possible Discriminator is:

$$D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_g(x)}$$

- The Equilibrium: When the Generator is perfect ($p_g = p_{data}$), the formula becomes:

$$D^*(x) = \frac{p_{data}}{p_{data} + p_{data}} = \frac{1}{2}$$

- Meaning: The Discriminator is now completely confused. It can no longer find any feature to distinguish between real and fake. It is reduced to 50/50 guessing.

Optimal Discriminator

- First, we rewrite the expectation formula as an integral over the data space x:

$$V(G,D) = \int_x p_{data}(x)\log(D(x)) + p_g(x)\log(1 - D(x))dx$$

- Since we want to maximize the integral, we can simply maximize the expression inside the integral for any given point x. Let's simplify the notation:
  - Let $a = p_{data}(x)$
  - Let $b = p_g(x)$
  - Let $y = D(x)$ (this is the value we want to find)
- Now we have a simple function to maximize:

$$f(y) = a\log(y) + b\log(1 - y)$$

Optimal Discriminator

- To find the maximum, we take the derivative with respect to y and set it to zero:

$$f'(y) = \frac{a}{y} - \frac{b}{1-y} = 0$$

- Solve for y

$$\frac{a}{y} = \frac{b}{1-y}$$

$$a(1-y) = by$$

$$a - ay = by$$

$$a = y(a+b)$$

$$y = \frac{a}{a+b}$$

- Substituting our original variables back in:

$$D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_g(x)}$$

Loss function

- If we assume the Discriminator is always "at its best" ($D = D^*$), we can plug that optimal value back into our original Value Function. Through some algebraic magic, the equation transforms into:

$$V(G, D^*) = 2 \cdot JSD(p_{data} \parallel p_g) - \log(4)$$

- What is JSD?

  The Jensen-Shannon Divergence is a “smooth” and "symmetrized" version of the KL Divergence. It is defined as:

$$JSD(P \parallel Q) = \frac{1}{2} KL(P \parallel M) + \frac{1}{2} KL(Q \parallel M)$$

  Where M is the average distribution: $M = \frac{1}{2}(P + Q)$

Why Does This Matter for Images?

- **Symmetry**: Unlike VAEs, where the order of P and Q changes the behaviour, JSD treats the "Real" and "Fake" distributions as equal partners.
- **Finite Limits**: JSD is bounded between 0 and $\log(2)$. This prevents the "exploding gradients" often seen in other metrics.
- **Sharpness**: Because JSD doesn't have the same "zero-avoiding" behaviour as the KL divergence used in VAEs, the Generator isn't forced to "cover" every single real data point with a blurry average. It can focus on creating high-quality, distinct samples.

Loss function derivation

- Substitute $D^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_g(x)}$ into the value function equation:

$$V(G, D^*) = \mathbb{E}_{x \sim p_{data}} \left[\log \frac{p_{data}(x)}{p_{data}(x) + p_g(x)} \right] + \mathbb{E}_{x \sim p_g} \left[\log \left(1 - \frac{p_{data}(x)}{p_{data}(x) + p_g(x)} \right) \right]$$

- Simplify the "Fake" part: $1 - \frac{p_{data}}{p_{data} + p_g} = \frac{p_{data} + p_g - p_{data}}{p_{data} + p_g} = \frac{p_g(x)}{p_{data}(x) + p_g(x)}$.

$$V(G, D^*) = \mathbb{E}_{x \sim p_{data}} \left[\log \frac{p_{data}(x)}{p_{data}(x) + p_g(x)} \right] + \mathbb{E}_{x \sim p_g} \left[\log \frac{p_g(x)}{p_{data}(x) + p_g(x)} \right]$$

- To turn these into KL Divergences, we need the denominator to be the average of the two distributions ($M = \frac{p_{data} + p_g}{2}$).
- To get that "2" in there, we multiply the top and bottom of the fractions by 2:

$$V(G, D^*) = \mathbb{E}_{x \sim p_{data}} \left[\log \frac{p_{data}(x)}{2 \cdot \frac{p_{data}(x) + p_g(x)}{2}} \right] + \mathbb{E}_{x \sim p_g} \left[\log \frac{p_g(x)}{2 \cdot \frac{p_{data}(x) + p_g(x)}{2}} \right]$$

Loss function derivation

- Using log rules ($\log \frac{A}{2B} = \log \frac{A}{B} - \log 2$), we pull out the $\log 2$ from both terms:

$$V(G, D^*) = \left(\mathbb{E}_{x \sim p_{data}} \left[\log \frac{p_{data}(x)}{M(x)} \right] - \log 2 \right) + \left(\mathbb{E}_{x \sim p_g} \left[\log \frac{p_g(x)}{M(x)} \right] - \log 2 \right)$$

- The definition of KL Divergence is $KL(P \parallel Q) = \mathbb{E}_{x \sim P}[\log \frac{P(x)}{Q(x)}]$.

$$V(G, D^*) = KL(p_{data} \parallel M) + KL(p_g \parallel M) - 2 \log 2$$

- By definition, $JSD(P \parallel Q) = \frac{1}{2} KL(P \parallel M) + \frac{1}{2} KL(Q \parallel M)$.
- Therefore, $2 \cdot JSD = KL(P \parallel M) + KL(Q \parallel M)$.
- Substituting that in (and noting that $2 \log 2 = \log 4$):

$$V(G, D^*) = 2 \cdot JSD(p_{data} \parallel p_g) - \log 4$$

Visualizing the Training: The Converging Distributions

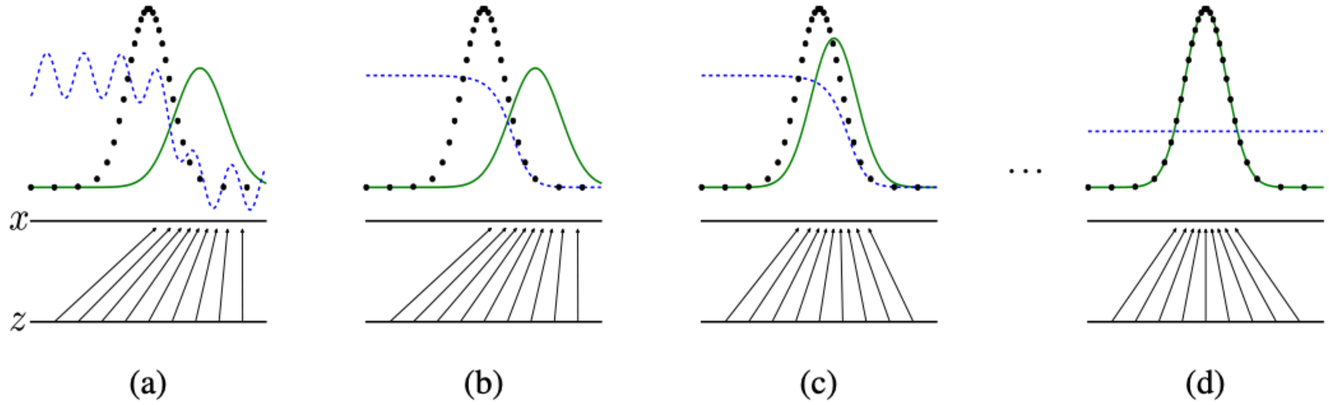


Figure 1: Generative adversarial nets are trained by simultaneously updating the **d**iscriminative distribution (D, blue, dashed line) so that it discriminates between samples from the data generating distribution (black, dotted line) $p_{\boldsymbol{x}}$ from those of the **g**enerative distribution p_g (G) (green, solid line). The lower horizontal line is the domain from which $\boldsymbol{z}$ is sampled, in this case uniformly. The horizontal line above is part of the domain of $\boldsymbol{x}$. The upward arrows show how the mapping $\boldsymbol{x} = G(\boldsymbol{z})$ imposes the non-uniform distribution p_g on transformed samples. G contracts in regions of high density and expands in regions of low density of p_g. (a) Consider an adversarial pair near convergence: p_g is similar to p_{data} and D is a partially accurate classifier. (b) In the inner loop of the algorithm D is trained to discriminate samples from data, converging to $D^*(\boldsymbol{x}) = \frac{p_{\text{data}}(\boldsymbol{x})}{p_{\text{data}}(\boldsymbol{x}) + p_g(\boldsymbol{x})}$. (c) After an update to G, gradient of D has guided $G(\boldsymbol{z})$ to flow to regions that are more likely to be classified as data. (d) After several steps of training, if G and D have enough capacity, they will reach a point at which both cannot improve because $p_g = p_{\text{data}}$. The discriminator is unable to differentiate between the two distributions, i.e. $D(\boldsymbol{x}) = \frac{1}{2}$.

Credit: Goodfellow et al. (2014) “Generative Adversarial Nets”

Why Does the Blue Line Flatten?

- As the green line (p_g) becomes identical to the black line (p_{data}), there is no longer any statistical difference between them. At that point:

$$D^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_g(x)} = \frac{p_{data}(x)}{2p_{data}(x)} = 0.5$$

- The Discriminator is now "guessing" because the forgery is perfect.

The Training Loop

- **Update the Discriminator** (D):
  - **Sample**: Grab a batch of real images (x) and a batch of noise (z).
  - **Forward Pass**:
    - $D(x)$: Score the real images (Target = 1).
    - $D(G(z))$: Score the fake images produced by the current G (Target = 0).
  - **Backprop**: Calculate the total loss and update only the weights of D.
  - **Goal**: Make D a better critic for this specific version of G.
- **Update the Generator** (G):
  - **Sample**: Grab a new batch of noise (z).
  - **Forward Pass**: Pass $G(z)$ into the Discriminator.
  - **The Trick**: Calculate the loss using the Label = 1 (as if the fakes were real).
  - **Backprop**: Flow the error from the Discriminator's output back into G. Crucially, freeze D's weights during this step.
  - **Goal**: Shift G's output to a region where D thinks the images are real.

Challenges in Training (The Dark Side of GANs)

- **Mode Collapse (The "One-Trick Pony")**
  - **The Problem**: The Generator discovers one specific type of image that the Discriminator currently finds "very real" (e.g., a really good looking '7' in the MNIST dataset).
  - **The Result**: Instead of learning to make all digits, the Generator maps every input z to that same '7'.
  - **The Loop**: The Discriminator eventually realizes all '7's are fake, then the Generator moves to a '1', and they circle each other forever without covering the whole dataset.
- **Vanishing Gradients**
  - **The Problem**: If the Discriminator is too good, its output is $0.000000...1$ for fakes.
  - **The Result**: Looking at the math of the Sigmoid function, the gradient becomes nearly zero in these "saturated" regions.
  - **The Consequence**: The Generator stops learning because its "teacher" is just giving a "0" grade without explaining why it failed.
- **Non-Convergence**
  - **The Problem**: Because G and D are playing a game, they can enter an infinite loop of "chasing" each other's weights (Oscillation) rather than settling down.

Non-Convergence

- In standard Deep Learning (like an Image Classifier), we use **Gradient Descent** to find a local minimum of a fixed loss surface. It's like finding the bottom of a valley.
- In GANs, we are looking for a **Nash Equilibrium** in a game between two players. This is much harder because:
  - **Moving Targets**: When the Generator (G) updates its weights to fool the Discriminator (D), it changes the "landscape" for D. When D updates to catch G, it changes the landscape for G.
  - **Lack of Objective Function**: There is no single "mountain" we are climbing. Instead, we have a "Min-Max" objective where the surface is constantly warping.
  - **Oscillations**: Instead of converging to a single point, the models often "chase" each other in circles. G learns to create a specific feature, D learns to reject it, G abandons that feature for a new one, and D follows.

Evaluation Metrics

- **The Challenge**: Standard loss functions (like MSE) don't work for GANs because we don't have a "target" image to compare against. We need a way to quantify two things:
  - **Fidelity**: Do the images look like clear, recognizable objects?
  - **Diversity**: Does the model generate a wide variety of objects, or just the same thing over and over?
- Instead we use the **following metrics**:
  - Inception Score (IS)
  - Frechet Inception Distance (FID)

Inception Score (IS)

- This metric uses a pre-trained image classifier (Google's **Inception v3**) to "look" at the generated images.
  - **Sharpness (Low Entropy)**: When the model sees a fake image, is it confident it's a specific class (e.g., "That is definitely a dog")?
  - **Diversity (High Entropy)**: Over a large batch of images, does the model generate many different classes (dogs, cars, trees), or just one?
  - The Goal: Higher is better.

Deep Dive: Inception Score (IS)

- The Inception Score relies on two mathematical properties of the output from a pre-trained **Inception v3** classifier:
- **Individual Quality (Conditional Probability $p(y|x)$):**
  - We feed one fake image x into the classifier.
  - If the image is sharp and recognizable, the classifier should output a high probability for **one specific class** (e.g., 0.99 for "Zebra").
  - We want the **entropy to be low** for a single image.
- **Global Diversity (Marginal Distribution $p(y)$):**
  - We feed a large batch (e.g., 50,000) of fake images.
  - We average all the label predictions. If the GAN is diverse, the average should be uniform (10% dogs, 10% planes, etc.).
  - We want the **entropy to be high** for the whole batch.
- The Formula:
  - Measure how different the "specific" label of one image is from the "average" labels of all images.

$$IS(G) = \exp(\mathbb{E}_{x \sim p_g}[KL(p(y|x) \parallel p(y))])$$

The Problems with IS

- While it was the first major metric, it has significant blind spots
- **"Internal" Mode Collapse**:
  - IS can't tell if you are generating 1,000 different Dalmatians or the exact same Dalmatian 1,000 times.
  - As long as it looks like a Dalmatian, the score stays high.
- **Classifier Bias:**
  - If your GAN generates something that isn't in the 1,000 ImageNet classes (like a specific medical X-ray), the Inception model won't know what to do with it, rendering the score meaningless.
  - IS is only guaranteed to work if the classes you are interested in is contained within ImageNet

The Problems with IS

- **No Comparison to Reality**:
  - IS only looks at the fake images.
  - It doesn't actually check if the fakes look like the specific real dataset you used.
- **Vulnerability to Artifacts:**
  - Sometimes, "adversarial noise" (random-looking pixels) can trick the Inception network into being very confident about a class, giving a high score to an image that looks like garbage to a human.
  - Needs constant human validation

Fréchet Inception Distance (FID)

- **The Problem with IS**: The Inception Score never actually looks at the real training data. It only cares if the fakes are recognizable.
- **The Solution**: FID measures how similar the distribution of generated images is to the distribution of real images.
- How it Works (The Feature Space):
  - We take a batch of **Real Images** and a batch of **Fake Images**.
  - We feed both through a pre-trained **Inception v3** network.
  - We "capture" the data at the **coding layer** (the last pooling layer before the final classification).
  - This layer represents high-level features (eyes, wheels, textures) rather than raw pixels.

The Math of FIS: Comparing Gaussians

- We represent the features of the real images as a Gaussian distribution with mean μ_r and covariance Σ_r. We do the same for the fakes (μ_g, Σ_g).
- The FID is the **Wasserstein-2 distance** between these two Gaussians:

$$d^2((\mu_r, \Sigma_r), (\mu_g, \Sigma_g)) = \|\mu_r - \mu_g\|^2 + Tr(\Sigma_r + \Sigma_g - 2(\Sigma_r\Sigma_g)^{1/2})$$

  - **Part 1** ($\|\mu_r - \mu_g\|^2$): Do the fakes have the same average features as the real images?
  - **Part 2** ($Tr(\ldots)$): Do the fakes have the same variety and correlations as the real images?
  - The Goal: Lower is better.

Why FID is Better than IS

- **Consistency**: FID is much more robust to noise.
- **Diversity Check**: If the Generator suffers from Mode Collapse (e.g., it only makes one type of dog), its covariance Σ_g will be very different from the real data's Σ_r, resulting in a bad (high) score.
- **Sensitivity**: If images are blurry or have weird artifacts, the Inception features will shift, and the FID will increase.

Summary — Mastering the GAN Framework

- **The Fundamental Game**
  - GANs are a Zero-Sum game where the Generator (G) creates and the Discriminator (D) critiques.
  - The equilibrium is reached when $D(x) = 0.5$, meaning the forgeries are indistinguishable from reality.
- **The Math of Success**
  - Under optimal conditions, the GAN objective minimizes the **Jensen-Shannon Divergence** (JSD).
  - Unlike VAEs, this metric doesn't force the model to be "blurry," leading to the **sharp, high-fidelity images** GANs are famous for.

Summary — Mastering the GAN Framework

- **Measures**
  - **Inception Score (IS)**: Focuses on "recognizable objects" and "class variety."
  - **FID**: It uses the **Trace** of covariance matrices to ensure the fake distribution matches the real distribution in feature space.
- **Hard Truths of Training**
  - **Instability**: The "rotational" gradients mean the models often orbit the solution rather than hitting it.
  - **Mode Collapse**: The risk of the Generator becoming a "one-trick pony" by ignoring the diversity of the dataset.